

# GEC: A Unified Framework for Interactive Decision Making in MDP, POMDP, and Beyond

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Joint work with  
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RL Theory Seminar

- 1 Overview
- 2 Problem Setup
- 3 Complexity Measure { GEC
- 4 Algorithm Design
- 5 Discussions

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1 Overview

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# Interactive Decision Making



The agent interacts with the unknown environment and aims to **maximize** its own reward.

Can we perform **sample-efficient** learning for interactive decision making?

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  - | Naive exploration incurs an **exponential** sample complexity (Kakade, 2003);
  - | Design algorithms with strategic exploration;

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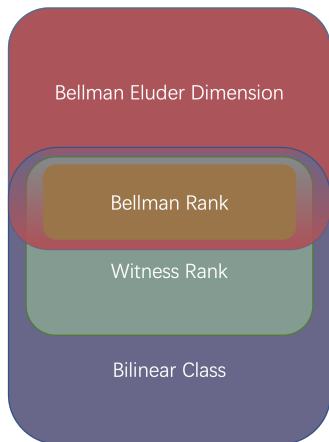
- Exploration-exploitation tradeo :
  - Naive exploration incurs an **exponential** sample complexity (Kakade, 2003);
  - Design algorithms with strategic exploration;
- Large state space:
  - $(\frac{P}{SAH^2T})$  lower bound for tabular RL (Jaksch et al., 2010);
  - Sample-efficient learning for RL with (general) function approximation;

Can we perform **sample-efficient** learning for interactive decision making?

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- Large state space:
  - |  $(\frac{P}{SAH^2T})$  lower bound for tabular RL (Jaksch et al., 2010);
  - | Sample-efficient learning for RL with (general) function approximation;
- Partial observations:
  - |  $(A^H)$  lower bound for general POMDPs (Krishnamurthy et al., 2016);
  - | Identify tractable partially observable RL models and design efficient algorithms.



# Previous Works



Fully Observable RL

Weakly Revealing POMDP

Latent MDP

Decodable POMDP

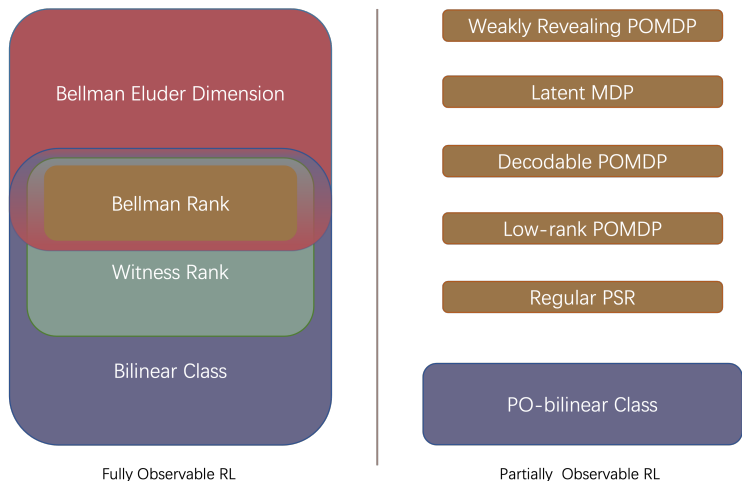
Low-rank POMDP

Regular PSR

PO-bilinear Class

Partially Observable RL

## Previous Works



1. Different complexity measures and algorithms;
2. Fully observable RL and partially observable RL are separate.

Propose a new complexity measure { Generalized Eluder Coefficient (GEC) } that can capture **nearly all** known tractable RL problems.

## Our Work

Algorithm:

Generic posterior sampling algorithm;

Generic UCB-based algorithm;

Maximize to explore (MEX) algorithm;

Proposed algorithms can be implemented in both **model-free** and **model-based** fashion, under both **fully observable** and **partially observable** settings.

## Our Work

Algorithm:

- Generic posterior sampling algorithm;
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Proposed algorithms can be implemented in both **model-free** and **model-based** fashion, under both **fully observable** and **partially observable** settings.

Theory:

The above three algorithms enjoy the regret of

$$\mathcal{O}(\text{poly}(d_{\text{GEC}}; H) T^{1=2}) \text{ or } \mathcal{O}(\text{poly}(d_{\text{GEC}}; H) T^{2=3});$$

These three algorithms can learn low GEC problems sample-efficiently;  
Match existing regret bounds for Bellman eluder dimension (Jin et al., 2021) and bilinear class (Du et al., 2021).

A **new** and **unified** understanding of both fully observable and partially observable RL.

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# Interactive Decision Making

Episodic Interactive Decision Making  $(O; A; H; P; R)$

$O$ : observation space;

$A$ : action space;

$H$ : length of each episode;

$P = \{P_h\}_{h \in [H]}$ :  $P_h(o_{h+1} | j, h)$  denotes the probability of generating the observation  $o_{h+1}$  given the history  $h = (o_{1:h}; a_{1:h})$ ;

$R = \{R_h\}_{h \in [H]}$ :  $R_h^+ : O \times A \rightarrow \mathbb{R}^+$ : reward functions;

Initial observation is sampled from a fixed distribution;

Assumption:  $\sum_{h=1}^H R_h = 1$ .

# Policy, Value Function, and Learning Objective

Policy  $\pi = \{ \pi_h \}_{h \in \mathcal{H}}$ :  $\pi_h : (\mathcal{O} \times \mathcal{A})^{h-1} \times \mathcal{O} \rightarrow \Delta(\mathcal{A})$  is a mapping from an observation-action sequence to a distribution over actions.

Visitation probability  $P(\pi) = \sum_{h=0}^H P_h(\pi)$ , where  $P_h(\pi)$  and  $\pi_h$  are defined by

$$P_h(\pi) = \prod_{h^0=1}^h \pi_{h^0}(a_{h^0} | j_{h^0-1}; o_{h^0})$$

Value function:

$$V := \mathbb{E} \sum_{h=1}^H r_h$$

Optimal policy:  $\pi^* = \arg\max_{\pi} V$ , optimal value:  $V^* = V(\pi^*)$ .

Learning objective: An online algorithm predicts  $\hat{g}_{t=1}^T$ , its regret is defined as

$$\text{Reg}(T) = \sum_{t=1}^T V - V^*$$



## Example 1: MDP

Episodic Markov Decision Process (MDP)  $(S; A; H; P; R)$

$$O = S \text{ and } P_h(x_{h+1} | x_{1:h}; a_{1:h}) = P_h(x_{h+1} | x_h; a_h);$$

Markov policy:  $\pi = f_h : S \rightarrow A$ ;

V-function and Q-function

$$V_h(x) := E_{x^0} \sum_{t=h}^{\infty} \gamma^t r_{h^0}(x_{h^0}; a_{h^0}) \quad x_h = x;$$

$$Q_h(x; a) := E_{x^0} \sum_{t=h}^{\infty} \gamma^t r_{h^0}(x_{h^0}; a_{h^0}) \quad x_h = x; a_h = a;$$

Optimal policy  $\pi^*$ , optimal Q-function  $Q^*$ ;

Bellman optimality equation:

$$Q_h(x; a) = (T_h Q_{h+1})(x; a) := r_h(x; a) + E_{x^0} \sum_{j \in \mathcal{X}} P_h(j|x;a) \max_{a^0 \in \mathcal{A}} Q_{h+1}(x^0; a^0);$$

Bellman residual:

$$E_h(Q; x; a) = Q_h(x; a) - (T_h Q_{h+1})(x; a);$$

## Example 2: POMDP

Episodic partially observable Markov decision process (POMDP)

$$(S; O; A; H; P; O = f O_h g_{h_2[H]}; R);$$

$$P_h(x_{h+1} | x_{1:h}; a_{1:h}) = P_h(x_{h+1} | x_h; a_h),$$

$O_h(o | x)$  is the probability of observing  $o$  at state  $x$  and step  $h$ ;

Learning POMDPs:

Negative Results:

- | exponential lower bound in the worst-case (Krishnamurthy et al., 2016);

Positive results:

- | Weakly revealing POMDPs (Jin et al., 2020):  $O = S$  and  $\min_{h_2[H]} \min(O_h) > 0$ ;
- | Decodable POMDPs (Du et al., 2019; Efroni et al., 2022):  $\exists$  unknown encoder  $h : O \rightarrow S$  such that  $h(O_h) = x_h$ ;
- | latent MDP with sufficient test (Kwon et al., 2021), low-rank POMDP (Wang et al., 2022), and regular PSR (Zhan et al., 2022).

# Function Approximation

General function approximation: hypothesis class  $\mathcal{H} = \{h_1, \dots, h_n\}$ ;

Model-based hypothesis  $\mathcal{H} = \{(\mathcal{P}_f; r_f) \mid \mathcal{P}_f \in \mathcal{P}, r_f \in \mathcal{R}\}$ ,

- |  $\pi_{h,f}$ : optimal policy corresponding to the model  $f$ ;
- |  $V_{h,f} = Q_{h,f}$ : optimal value/Q function corresponding to the model  $f$ ;
- |  $f$ : true model;  $V_{h,f} = V_h, Q_{h,f} = Q_h$ ;

Value-based hypothesis (for MDP)  $\mathcal{H} = \{Q_{h,f}, g_{h,f} \mid Q_{h,f} \in \mathcal{Q}, g_{h,f} \in \mathcal{G}\}$ ;

- |  $\pi_{h,f}(\cdot) = \operatorname{argmax}_{a \in \mathcal{A}} Q_{h,f}(\cdot; a)$ ;
- |  $V_{h,f}(\cdot) = \max_{a \in \mathcal{A}} Q_{h,f}(\cdot; a)$ ;
- |  $f = Q$ ;

Realizability assumption:  $f \in \mathcal{H}$ .

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# Motivation

By the value decomposition lemma (Jiang et al., 2017), we have

$$\sum_{t=1}^T \mathbb{E} \left[ \sum_{h=1}^H \left( \mathbb{E}_{f^t; x_h^t; a_h^t} \left\{ Z_{f^t} \right\} \right) \right] = \sum_{t=1}^T \mathbb{E}_{f^t} \left[ \sum_{h=1}^H \left( \mathbb{E}_h \left\{ Z_{f^t} \right\} \right) \right] + \sum_{t=1}^T \mathbb{E} \left[ \sum_{h=1}^H \left( \mathbb{E}_{f^t; x_h^t; a_h^t} \left\{ Z_{f^t} \right\} \right) \right]$$

$\underbrace{\hspace{10em}}_{\text{Reg}(T)} \qquad \underbrace{\hspace{10em}}_{\text{Bellman residual}} \qquad \underbrace{\hspace{10em}}_{\text{bias}}$

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 = \underbrace{\sum_{t=1}^T \mathbb{E} \left[ \sum_{h=1}^H \left( \mathbb{E}_{f^t; x_h^t; a_h^t} \left\{ \underbrace{Z}_{\text{Bellman residual}} \right\} \right) \right]}_{\text{Reg}(T)} + \underbrace{\sum_{t=1}^T \mathbb{E} \left[ \sum_{h=1}^H \left( \mathbb{E}_{f^t} \left\{ \underbrace{Z}_{\text{bias}} \right\} \right) \right]}_{\text{bias}}
 \end{aligned}$$

(if  $V = V_{f^t}$ )

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(if  $V = V_{f^t}$ )

UCB-based algorithm:  $f^t = \arg\max_{f \in \text{confidence set}} V_f$  to ensure **optimism**;

# Motivation

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$$\sum_{t=1}^T \mathbb{E} \left[ \sum_{h=1}^H V_{f^t} - \min_{f \in \mathcal{F}} \sum_{h=1}^H V_f \right] = \sum_{t=1}^T \mathbb{E} \left[ \sum_{h=1}^H \underbrace{f^t(x_h^t; a_h^t) - \min_{f \in \mathcal{F}} f(x_h^t; a_h^t)}_{\text{Bellman residual}} \right] + \sum_{t=1}^T \mathbb{E} \left[ \sum_{h=1}^H \underbrace{V_{f^t} - V_{f^t}}_{\text{bias}} \right]$$

(if  $V = V_{f^t}$ )

UCB-based algorithm:  $f^t = \arg\max_{f \in \text{confidence set}} V_f$  to ensure **optimism**;

"Mismatch" between **Goal** and **Guarantee**:

- Goal:**  $f^t$  performs well on the **unseen data**  $t$ ;

$$\sum_{h=1}^H \mathbb{E}_{f^t} \left[ \mathbb{E}_h \left[ f^t(x_h^t; a_h^t) - \min_{f \in \mathcal{F}} f(x_h^t; a_h^t) \right] \right] \text{ is small?}$$

- Guarantee:**  $f^t$  is good on the **historical dataset**  $f^1; f^2; \dots; f^{t-1}$ ;

$$\sum_{h=1}^H \sum_{s=1}^{t-1} \mathbb{E}_{f^s} \left[ \mathbb{E}_h \left[ (f^t(x_h^s; a_h^s) - f^s(x_h^s; a_h^s))^2 \right] \right] \text{ is small}$$



# Challenge

Connect the **Goal** and **Guarantee** "generalization" from the past to the future:

- Goal:  $f^t$  performs well on the **unseen data**  $t$ ;

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In supervised learning  $\mathcal{Z}_s, g_{s=1}^t$  and an unseen  $\mathcal{Z}^t$  are i.i.d. sampled from a fixed distribution  $D_{\text{data}}$ ;

- Reliability + low hypothesis complexity (e.g., covering number) ensure generalization;

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$h=1$

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$$\mathcal{X}^t \quad \mathcal{X}^1 \quad E_{f^s} E_h (f^t; x_h^s; a_h^s)^2 \quad \text{is small}$$

$h=1 \quad s=1$

In supervised learning,  $z_s, g_{s=1}^t$  and an unseen  $\mathcal{Z}^t$  are i.i.d. sampled from a fixed distribution  $D_{\text{data}}$ ;

- Reliability + low hypothesis complexity (e.g., covering number) ensure generalization;

In RL,  $\mathcal{X}^1, \mathcal{X}^2, \dots, \mathcal{X}^t$ , distribution shift exists all the time!

Require an additional structure assumption permits "generalization" from the past to the future (in an online manner).

# Simplified Generalized Eluder Coefficient

Generalized Eluder Coefficient (GEC) is the smallest  $d_{\text{GEC}}$  such that

$$\sum_{t=1}^T \sum_{h=1}^H E_{f^t} E_h(f^t; x_h^t; a_h^t) \leq d_{\text{GEC}} \sum_{h=1}^H \sum_{t=1}^T \sum_{s=1}^S E_{f^s} E_h(f^t; x_h^s; a_h^s)^2$$

Goal: prediction error
Guarantee: training error

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- On average, if  $f^t \in H$  is consistent with the historical data, then the prediction error on unseen  $t$ -th trajectory would also be small (but is amplified by GEC);

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- On average, if  $f^t \in H$  is consistent with the historical data, then the prediction error on unseen  $t$ -th trajectory would also be small (but is amplified by GEC);
- Optimism ( $V - V_{f^t}$ ) + low GEC + small training error = low-regret learning:

$$\text{Reg}(T) = \sum_{t=1}^T \sum_{h=1}^H E_{f^t} E_h (f^t; x_h^t; a_h^t) \leq d_{\text{GEC}} \sum_{h=1}^H \sum_{t=1}^T \sum_{s=1}^S \underbrace{E_{f^s} E_h (f^s; x_h^s; a_h^s)^2}_{\text{training error}} \quad (1=2) \quad p \frac{\text{training error}}{d_{\text{GEC}} H T}$$

- For LinUCB (Chu et al., 2011), UCRL2 (Jaksch et al., 2010), UCRL-VTR (Ayoub et al., 2020), GOLF (Jin et al., 2021)... only has a logarithmic dependency in  $T$ .

# Generalized Eluder Coefficient

$$\sum_{t=1}^T V_{f^t} = \sum_{t=1}^T \sum_{h=1}^H \underbrace{E_{f^t} E_h(f^t; x_h; a_h)}_{\text{Goal: prediction error}} \cdot d_{\text{GEC}} \sum_{h=1}^H \sum_{t=1}^T \sum_{s=1}^S \underbrace{E_{f^s} E_h(f^t; x_h; a_h)^2}_{\text{Guarantee: training error}}^{1=2}.$$

# Generalized Eluder Coe cient

$$\sum_{t=1}^T V_{f^t} - V_{f^*} = \sum_{t=1}^T \underbrace{E_{f^t}^h}_{\text{Goal: prediction error}} + \sum_{h=1}^d \sum_{t=1}^T \underbrace{E_{f^*}^h}_{\text{Guarantee: training error}} \sum_{s=1}^{t-1} E_{f^s}^h E_h(f^t; x_h; a_h)^2 \quad (1=2)$$

## Definition (Generalized Eluder Coe cient)

Given a hypothesis class  $\mathcal{H}$ , a discrepancy function  $\eta = f - g_{\mathcal{H}}$ , an exploration policy class  $\text{exp}$ , the generalized eluder coe cient  $\text{GEC}(\mathcal{H}; \eta; \text{exp}; \epsilon)$  is the smallest  $d \geq 0$  such that for any sequence of hypotheses and exploration policies

$f^t; f_{\text{exp}}(f^t; h) g_{\mathcal{H}}[H] g_{\mathcal{H}}[T]$ :

$$\sum_{t=1}^T \underbrace{V_{f^t} - V_{f^*}}_{\text{prediction error}} + d \sum_{h=1}^d \sum_{t=1}^T \underbrace{E_{\text{exp}(f^t; h)} \eta^2(f^t; h)}_{\text{training error}} + 2 \sum_{t=1}^T \underbrace{p \overline{\text{dHT}}_{\{Z\}} + \text{HT}}_{\text{burn-in cost}} \quad (1=2)$$

Flexible choices of discrepancy functions and exploration policies.

The GEC captures the hardness of exploration-exploitation trade-off by comparing the **out-of-sample prediction error** with the **in-sample training error**;



# Generalized Eluder Coefficient: MDP Examples

Q-type problems :

$$\sum_{t=1}^T V_{f^t} - V_{f^*} \leq d_Q \sum_{h=1}^H \sum_{t=1}^T \sum_{s=1}^{1=2} E_{f^s} E_h(f^t; x_h; a_h)^2$$

V-type problems:

$$\sum_{t=1}^T V_{f^t} - V_{f^*} \leq d_V \sum_{h=1}^H \sum_{t=1}^T \sum_{s=1}^{1=2} E_{f^s \sim \text{Unif}(A)} E_h(f^t; x_h; a_h)^2$$

where  $f^s \sim \text{Unif}(A)$  means executing  $f^s$  for the first  $h-1$  steps and then take a random  $a_h \in A$ .

Model-based problems:

$$\sum_{t=1}^T V_{f^t} - V_{f^*} \leq d \sum_{h=1}^H \sum_{t=1}^T \sum_{s=1}^{1=2} E_{\sim} D_H^2(P_{h,f^t}(\cdot | x_h; a_h); P_{h,f^*}(\cdot | x_h; a_h))$$

where  $\sim$  is either  $f^s$  (Q-type) or  $f^s \sim \text{Unif}(A)$  (V-type) and  $D_H^2(P; Q) = \frac{1}{2} E_{x \sim P} [(\frac{dQ(x)}{dP(x)} - 1)^2]$  is the Hellinger divergence.

## Relationship with Existing Complexity Measures

Bellman eluder dimension:

GEC  $\mathcal{O}(Hd_Q)$  Q-type;    GEC  $\mathcal{O}(AHd_V)$  V-type;

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GEC  $\mathcal{O}(Hd_Q)$  Q-type;      GEC  $\mathcal{O}(AHd_V)$  V-type;

Bilinear class:

GEC  $\mathcal{O}(Hd_{bil})$  Q-type;      GEC  $\mathcal{O}(AHd_{bil})$  V-type;

## Relationship with Existing Complexity Measures

Bellman eluder dimension:

GEC  $\mathcal{O}(\text{Hd}_Q)$  Q-type;      GEC  $\mathcal{O}(\text{AHd}_V)$  V-type;

Bilinear class:

GEC  $\mathcal{O}(\text{Hd}_{\text{bil}})$  Q-type;      GEC  $\mathcal{O}(\text{AHd}_{\text{bil}})$  V-type;

Witness rank:

GEC  $\mathcal{O}(\text{Hd}_Q = \frac{2}{\text{wit}})$ ; Q-type;      GEC  $\mathcal{O}(\text{AHd}_V = \frac{2}{\text{wit}})$ ; V-type;

# Relationship with Existing Complexity Measures

GEC (model-based POMDP version):

$$d_{\text{GEC}} = \sum_{t=1}^T \sum_{h=0}^{H-1} \sum_{s=1}^S \left( \sum_{f \in \mathcal{F}} V_f^t - V^t \right) \left( \sum_{f \in \mathcal{F}} P_f^{\text{exp}(f^S; h)} - P_f^{\text{exp}(f^S; h)} \right);$$

where  $\text{exp}(f^S; h) := \sum_{f \in \mathcal{F}} P_f^{\text{exp}(f^S; h)} \text{Unif}(A)$

- revealing POMDPs:

$$d_{\text{GEC}} = \mathcal{O}(\text{poly}(S; A; H; 1/\epsilon));$$

- Decodable POMDPs:

$$d_{\text{GEC}} = \mathcal{O}(\text{poly}(S; A; H));$$

<sup>1</sup>Independent works Liu et al. (2022); Chen et al. (2022) identify similar PSR classes with regular conditions on observable operators (Jaeger, 2000).

# Relationship with Existing Complexity Measures

GEC (model-based POMDP version):

$$\sum_{t=1}^T V_{f^t} - V^* \leq d_{\text{GEC}} \sum_{t=1}^T \sum_{h=0}^{H-1} \sum_{s=1}^S \mathbb{E}_{\mathbf{x}^t} \left[ \mathbb{E}_{\mathbf{h}^t} \left[ \mathbb{E}_{\mathbf{s}^t} \left[ D_H^2 P_{f^t}^{\exp(f^s; h)}; P_f^{\exp(f^s; h)} \right] \right] \right] \quad 1=2;$$

where  $\mathbb{E}_{\exp(f^s; h)} := \mathbb{E}_{f^s} \mathbb{E}_h \text{Unif}(A) \quad \mathbb{E}_H \text{Unif}(A)$ .

- revealing POMDPs:

$$\text{GEC} \leq \mathcal{O}(\text{poly}(S; A; H; 1/\epsilon));$$

- Decodable POMDPs:

$$\text{GEC} \leq \mathcal{O}(\text{poly}(S; A; H));$$

-generalized regular PSR (new<sup>1</sup>)

- Impose some regular condition on the observable operator representation (Jaeger, 2000) of PSR.
- Nearly all known tractable POMDPs satisfy this regular condition;
- With proper exploration policies:

$$\text{GEC} \leq \mathcal{O}(\text{poly}(\text{complexity of PSR}; H; A; 1/\epsilon))$$

<sup>1</sup>Independent works Liu et al. (2022); Chen et al. (2022) identify similar PSR classes with regular conditions on observable operators (Jaeger, 2000).

# Summary of Relationships

GEC captures **nearly all** known tractable RL problems.

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# Algorithmic Design to Use GEC

$$\text{Reg}(T) = \sum_{t=1}^T V \quad V^{f^t} = \sum_{t=1}^T \underbrace{V_{f^t}}_{\text{prediction error}} + \sum_{t=1}^T \underbrace{V_{f^t}}_{\text{bias}}$$

$$d_{\text{GEC}} = \sum_{h=1}^H \sum_{t=1}^T \sum_{s=1}^1 \mathbb{E}_{\exp(f^s; h)} [f^s(f^t; h)] + \sum_{t=1}^T \underbrace{V_{f^t}}_{\text{bias}} :$$

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- How to control the training error?

- 1 The training error term is not available to the executed algorithm, e.g., the Bellman operator, or the true transition kernel  $P_f$  ;
  - 1 We need to approximate the training error by some loss functions and design effective estimation to achieve a low training error.

# Algorithmic Design to Use GEC

$$\text{Reg}(T) = \sum_{t=1}^T V_{f^t} - V_{f^*} = \sum_{t=1}^T \underbrace{V_{f^t} - V_{f^*}}_{\text{prediction error}} + \sum_{t=1}^T \underbrace{V_{f^*}}_{\text{bias}}$$

$$d_{\text{GEC}} = \sum_{h=1}^H \sum_{t=1}^T \mathbb{E} \left[ \sum_{s=1}^1 \underbrace{V_{f^s} - V_{f^*}}_{\text{training error}} \right] + \sum_{t=1}^T \underbrace{V_{f^*}}_{\text{bias}} :$$

- How to control the training error?

- The training error term is not available to the executed algorithm, e.g., the Bellman operator, or the true transition kernel  $P_f$  ;
  - We need to approximate the training error by some loss functions and design effective estimation to achieve a low training error.

- How to control the bias term?

- UCB-based algorithms directly have  $V_{f^*} - V_{f^*} = 0$
  - For other algorithms such as posterior sampling,  $V_{f^*} - V_{f^*} = 0$  is not directly available.

# A Generic Posterior Sampling Framework

## Posterior sampling algorithm

**Optimistic prior (Zhang, 2022):** Choose the prior that favors the hypotheses with higher values

$$p^0(f) \propto \exp(V_f); \quad > 0:$$

**Loss function:** Let

$$L_h^t(f) = L_h(f; \{f^s\}_{s=1}^t; \{D_h^s\}_{s=1}^t)$$

be a proxy of the unknown training error  $\mathbb{P}_{s=1}^t \mathbb{E}_{\exp(f^s; h)} [f^s(f; h)]$ .

**Posterior:**

$$p^t(f) / p^0(f) \propto \exp\left(V_f + \sum_{h=1}^H L_h^t(f)\right); \quad f^t \sim p^t(\cdot)$$

**Data collection:** For any  $h \in [H]$ , execute  $\exp(f^t; h)$  for  $N_{\text{batch}}$  times and collect samples  $D_h^t$ .

# Choices of Loss Functions (Model-free case)

Double sampling issue of model-free MDP (Antos et al., 2008):

$$\mathbb{E} \left[ \underbrace{s [Q_{h;f}(x_h^s; a_h^s) - r_h^s - V_{h+1;f}(x_{h+1}^s)]^2}_{\text{TD error}} \right] = \underbrace{\mathbb{E} \left[ s [E_h(f; x_h^s; a_h^s) - Z]^2 \right]}_{\text{Goal: training error}} + \underbrace{\frac{2}{|Z|}}_{\text{Sampling variance}} \mathbb{E} [r_h^s - V_{h+1;f}(x_{h+1}^s)]^2$$

1 Minimax formulation (GOLF (Jin et al., 2021), Conditional PS (Dann et al., 2021))<sup>2</sup>

$$L_h^t(f) = \sum_{s=1}^T [Q_{h;f}(x_h^s; a_h^s) - r_h^s - V_{h+1;f}(x_{h+1}^s)]^2$$

$$\log \mathbb{E}_{f_h, p_h^0(\cdot)} \exp \sum_{s=1}^T [Q_{h;f}(x_h^s; a_h^s) - r_h^s - V_{h+1;f}(x_{h+1}^s)]^2 ;$$

- ‡ The introduced log term cancels the variance;
- ‡ The log term requires **completeness** to deal with;

2 Trajectory average with  $N_{\text{batch}}$  i.i.d. data (OLIVE (Jiang et al., 2017), BiLin-UCB (Du et al., 2021))

$$L_h^t(f) = \sum_{s=1}^T \frac{1}{N_{\text{batch}}} \sum_{i=1}^{N_{\text{batch}}} [Q_{h;f}(x_{i;h}^s; a_{i;h}^s) - r_{i;h}^s - V_{h+1;f}(x_{i;h+1}^s)]^2 ;$$

- ‡ Sample mean admits a smaller variance:  $\text{Var}[X_m] = \frac{1}{m} \text{Var}[X]$ .

<sup>2</sup>Also used in some works on offline RL (Antos et al., 2008; Chen and Jiang, 2019).

## Choices of Loss Function (Model-based case)

- For MDPs, we choose

$$L_h^t(f) = \sum_{s=1}^t \log P_{h;f}(x_{h+1}^s | x_h^s; a_h^s);$$

where  $D_h^s = (x_h^s; a_h^s; x_{h+1}^s)$  is the tuple induced by  $\exp(f^s; h)$ .

- For POMDPs and PSRs, we choose

$$L_h^t(f) = \sum_{s=1}^t \log P_f(\overset{s}{h});$$

where  $D_h^s = \overset{s}{h}$  is the trajectory induced by  $\exp(f^s; h)$ .

## UCB Algorithm

Given the past  $t - 1$  iterations, we maintain a confidence set  $H_t \subseteq H$  such that  $f \in H_t$  with high probability;

Choose the most optimistic hypothesis  $f^t$ :

$$f^t = \operatorname{argmax}_{f \in H_t} V_f$$

Execute exploration policies  $f_{\text{exp}}(f^t; h) g_{h \in [H]}$  to collect data

## UCB Algorithm

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- Extend previous UCB algorithms (LinUCB, UCRL2, UCRL-VTR, GOLF, BiLinUCB, OMLE, ...) to a more general class (problems with low GEC);
- Theoretical analysis is relatively simple and well-understood;
- Hard to implement: need to solve a **constrained** optimization problem



## Maximize to Explore

Given the past  $t - 1$  iterations, we choose a proper loss  $L_h^{t-1}(f)$ ;

Choose the hypothesis  $f^t$ :

$$f^t = \underset{f}{\operatorname{argmax}} \left( V_f \sum_{h=1}^H L_h^{t-1}(f) \right)$$

An optimistic modification of loss minimization problem.

Execute exploration policies  $f_{\text{exp}}(f^t; h)g_{h2[H]}$  to collect data

Easy to implement: only need to optimize an **unconstrained** objective.

# Summary of Algorithm Design

$$\text{Reg}(T) = \sum_{t=1}^T V_{f^t} = \sum_{t=1}^T V_{f^t} + \sum_{t=1}^T V_{f^t} - \sum_{t=1}^T V_{f^t}$$

$$d_{\text{GEC}} = \sum_{h=1}^H \sum_{t=1}^T \mathbb{E}_{\exp(f^s; h)} \{ \underbrace{f^s(f^t; h)}_{\text{training error}} \} + \sum_{t=1}^T \underbrace{\left( V_{f^t} - \mathbb{E}_{\exp(f^s; h)} \{ V_{f^t} \} \right)}_{\text{bias}}$$

- How to control the training error?
  - | Choose proper loss functions to approximate the training error.
  - | Choose proper exploration policies to collect data.
- How to control the bias term?
  - | Optimistic posterior sampling
  - | UCB-based algorithm
  - | Maximize to explore (MEX)

## Theorem ((Zhong et al., 2022; Liu et al., 2023))

The above three algorithms enjoy the following regret bounds:

### 1 Value-based approach for MDPs

- Minimax formulation with **Realizability** + **Completeness**:  $\mathcal{O} \sqrt{d_{\text{GEC}} H T \log j H j}$  ;
- Trajectory average with **Realizability**:  $\mathcal{O} (d_{\text{GEC}}^2 H \log j H j)^{1/3} T^{2/3}$  <sup>a</sup>;

### 2 Model-based approach for MDP, POMDP, and PSR:

- **Realizability**:  $\mathcal{O} \sqrt{d_{\text{GEC}} H T \log j H j}$  .

<sup>a</sup>Also holds for PO-bilinear class.

- Interactive decision making with low GEC is learnable.
- Matches existing bound for Bellman eluder dimension (Jin et al., 2021) and Bilinear class (Du et al., 2021).

Optimistic modification + Low GEC + Effective training error estimation  
Sample-efficient learning.

# Table of Contents

- 1 Overview
- 2 Problem Setup
- 3 Complexity Measure { GEC
- 4 Algorithm Design
- 5 Discussions

## Similarities:

- Universality: subsume most of the known tractable RL problems;
- Reduction-based idea: convert regret minimization to new target;

## Differences:

- Different reduction ideas: in-sample estimation v.s. online learning;
- Different policy selection strategies: simple strategy v.s. minimax subroutine;
- Algorithm design:
  - ┆ GEC: flexible in algorithmic design: i) Posterior sampling, ii) UCB-based algorithm, and iii) Maximize to explore;
  - ┆ DEC: restrictive algorithm design: Estimation to decision-making (E2D);
- Regret upper bound:
  - ┆ GEC: match the best-known results;
  - ┆ DEC: suboptimal  $T^{3=4}$  regret bound (Foster et al., 2022) for bilinear class;
- Lower bound: DEC also characterizes the lower bound of the RL problems.

## Conclusion

New complexity measure  $\{ \text{GEC} \}$  that can capture nearly all known tractable interactive decision making problems.

reduce the out-of-sample prediction error to the in-sample training error.

Three efficient algorithms for interactive decision making with low GEC.

optimistic modification + an effective sequential estimation of training error.

A new and unified understanding for both fully observable and partially observable RL.

# Thank you!

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# Backup Slides

## Definition ( $\rho$ -independence between distributions)

Let  $G$  be a function class defined on  $X$ , and  $\mu_1, \dots, \mu_n$  be probability measures over  $X$ . We say  $\rho$  is  $\rho$ -independent of  $f_1, \dots, f_n, g$  with respect to  $G$  if there exists  $g \in G$  such that  $\frac{1}{n} \sum_{i=1}^n (\mathbb{E}_{\mu_i}[g])^2 > \rho$  but  $\mathbb{E}_{\mu_j}[g] \leq \rho$ .

The distributional eluder dimension  $\text{dim}_{\text{DE}}(G; \rho)$  is the length of the longest sequence  $f_1, \dots, f_n, g$  such that there exists  $\rho$  with  $\rho$  being  $\rho$ -independent of  $f_1, \dots, f_{i-1}, g$  for all  $i \in [n]$ .

- Let  $(I - T_h)H := f(x; a) - (f_h - T_h f_{h+1})(x; a) : f \in H, g$ ,  
 $(I - T_h)V_H := f(x) - (f_h - T_h f_{h+1})(x; f_h(x)) : f \in H, g$  be the set of Q/V type Bellman residuals induced by  $H$  at step  $h$ ;
- The Q/V-type  $\rho$ -BE dimension of  $H$  with respect to  $\rho$  is defined as

$$d_Q = d_V := \max_{h \in [H]} \dim_{\text{DE}}(I - T_h)H = \dim_{\text{DE}}(I - T_h)V_H; \rho, h; \rho$$

- We have  $\text{GEC} \leq \mathcal{O}(H d_Q)$  and  $\text{GEC} \leq \mathcal{O}(A H d_V)$ .

## Definition (Bilinear Class)

We say the RL problem is in a Bilinear class if there exist functions  $W_h : H \rightarrow V$  and  $X_h : H \rightarrow V$  for a Hilbert space  $V$ , such that  $\forall f \in H$  and  $h \in [H]$ , we have

$$\begin{aligned} E_f E_h(f; x_h; a_h) &= \langle W_h(f), X_h(f) \rangle \\ E_{x_h} E_{f; a_h} [I(f; h)] &= \langle W_h(g), X_h(f) \rangle \end{aligned}$$

where  $I$  is a loss function with  $x_h = (x_h; a_h; r_h; x_{h+1})$  and  $\sim$  is either  $\sim_f$  (Q-type) or  $\sim_g$  (V-type). The complexity of a bilinear class is characterized by the information gain:

$$\mathcal{I}(\cdot; X) = \sum_{h=1}^H \mathcal{I}(\cdot; X_h) \text{ with } X_h = \{X_h(f) : f \in H\}.$$

- With  $\tilde{f}^0(f; x_h; a_h) = \sqrt{E_{x_{h+1} | x_h; a_h} I(f; h)^2}$ , we have

$$\text{GEC} \leq \mathcal{I}(\cdot; X) \quad \text{Q-type}; \quad \text{GEC} \leq 2A \mathcal{I}(\cdot; X); \quad \text{V-type:}$$

## Definition (Q-type/V-type Witness Rank)

Given a discriminator class  $V = fV_h : S \times A \times S \rightarrow [0;1]^{g_{h2[H]}}$ . We say an MDP has witness rank  $d$  if given two models  $f; g \in H$ , there exists  $X_h : H \rightarrow \mathbb{R}^d$  and  $W_h : H \rightarrow \mathbb{R}^d$  such that

$$\max_{v \in V_h} E_{x_h \sim f; a_h \sim P_{h;g}(j_{x_h; a_h})} [E_{x^0 \sim P_{h;g}(j_{x_h; a_h})} v(x_h; a_h; x^0) - E_{x^0 \sim P_{h;f}(j_{x_h; a_h})} v(x_h; a_h; x^0)] \\ \leq \|W_h(g) - X_h(f)\|;$$

$$\text{wit}_{\sim} E_{x_h \sim f; a_h \sim P_{h;g}(j_{x_h; a_h})} [E_{x^0 \sim P_{h;g}(j_{x_h; a_h})} V_{h+1;g}(x^0) - E_{x^0 \sim P_{h;f}(j_{x_h; a_h})} V_{h+1;g}(x^0)] \\ \leq \|W_h(g) - X_h(f)\|;$$

where  $\sim$  is either  $\sim_f$  (Q-type) or  $\sim_g$  (V-type), and  $\text{wit}_{\sim} \in [0;1]$ .

- With details as in the model-based examples, we have

$$\text{GEC}_{\sim_f} 4d_Q H \log\left(\frac{1}{\text{wit}_{\sim_f}}\right) = \frac{2}{\text{wit}_{\sim_f}}; \quad \text{Q-type};$$

$$\text{GEC}_{\sim_g} 4d_V A H \log\left(\frac{1}{\text{wit}_{\sim_g}}\right) = \frac{2}{\text{wit}_{\sim_g}}; \quad \text{V-type};$$

## Example 3: Predictive State Representations (PSR)

### Predictive State Representation (PSR)

- History  $h = (o_{1:h}; a_{1:h}) = (o_1; a_1; \dots; o_h; a_h)$ ;
- Test (future)  $t_{h+1} = (o_{h+1:h+W}; a_{h+1:h+W-1})$ , where length  $W \geq N^+$ ;
- System dynamics matrix  $D_h$ : i) tests as rows and histories as columns; and ii) the  $(t_{h+1}; h)$ -th entry of  $D_h$  is equal to  $P(t_{h+1} | h)$ ;
- PSR rank  $d_{\text{PSR}}$ :  $d_{\text{PSR}} = \max_{h \in \mathcal{H}} \text{Rank}(D_h) = d_{\text{PSR}; h}$ ;
- Observable Operator Representation (Jaeger, 2000): given a PSR with a core test set  $\mathcal{F} \cup \mathcal{G}_{h \in \mathcal{H}}$ , there exists a set of matrices  $\mathbf{M}_h(o; a) \in \mathbb{R}^{|\mathcal{U}_{h+1}| \times |\mathcal{U}_h|}$ ,  $\mathbf{g}_{o \in \mathcal{O}; a \in \mathcal{A}; h \in \mathcal{H}}$ ;  $\mathbf{q}_0 \in \mathbb{R}^{|\mathcal{U}_1|}$  that can characterize its dynamics:

$$P(h) = \mathbf{M}_h(o_h; a_h) \mathbf{M}_{h-1}(o_{h-1}; a_{h-1}) \dots \mathbf{M}_1(o_1; a_1) \mathbf{q}_0$$

### Connection with POMDP

- $d_{\text{PSR}} = S$ :  $D_h = [P(t_{h+1} | h)] = [P(t_{h+1} | s_{h+1})] \quad [P(s_{h+1} | h)]$
- For one step revealing/decodable POMDPs, we can choose  $U_h = \mathcal{O}$

$$\mathbf{M}_h(o_h; a_h) = \begin{matrix} \mathcal{O}_{h+1} & \mathcal{T}_{h;a_h} & \text{diag}(\mathcal{O}_h(o_h; j)) & \mathcal{O}_h^y \\ \begin{matrix} | \\ \mathcal{R}^{\mathcal{O}} \end{matrix} \begin{matrix} \mathcal{Z} \\ \mathcal{S} \end{matrix} & \begin{matrix} | \\ \mathcal{R}^{\mathcal{S}} \end{matrix} \begin{matrix} \mathcal{Z} \\ \mathcal{S} \end{matrix} & \begin{matrix} | \\ \mathcal{R}^{\mathcal{S}} \end{matrix} \begin{matrix} \mathcal{Z} \\ \mathcal{S} \end{matrix} & \begin{matrix} | \\ \mathcal{R}^{\mathcal{S}} \end{matrix} \begin{matrix} \mathcal{O} \\ \mathcal{O} \end{matrix} \end{matrix}; \quad \mathbf{q}_0 = \mathcal{O}_{1-1} \in \mathbb{R}^{\mathcal{O}}$$

# Relationship with Existing Complexity Measures

## Definition ( $\rho$ -Generalized Regular PSR)

1. For any  $h \in [H]$  and  $\mathbf{x} \in \mathbb{R}^{J^{U_h}}$ , it holds that

$$\max_{o_{h:H}; a_{h:H}} \sum_{j \in U_h} \mathbf{M}_H(o_{h:H}; a_{h:H}) \mathbf{M}_h(o_h; a_h) \mathbf{x}_j \leq \frac{\rho \|\mathbf{x}\|_1}{k_1};$$

where  $(o_{h:H}; a_{h:H}) \in (O \times A)^{H-h+1}$ .

2. For any  $h \in [H-1]$  and  $\mathbf{x} \in \mathbb{R}^{J^{U_h}}$ , it holds that

$$\max_{(o_h; a_h) \in O \times A} \sum_{j \in U_{A;h+1}} \mathbf{M}_h(o_h; a_h) \mathbf{x}_j \leq \frac{\rho \|\mathbf{x}\|_1}{k_1};$$

where  $U_{A;h+1}$  is the the action sequences in the core test set  $U_{h+1}$ .<sup>a</sup>

<sup>a</sup>Independent works Liu et al. (2022); Chen et al. (2022) identify similar PSR classes with regular conditions on observable operators.

- Any revealing POMDP is an  $\rho = \frac{1}{S}$ -generalized regular PSR.
- Any decodable POMDP is a 1-generalized regular PSR.
- Latent MDPs with the full-rank test, low-rank POMDPs, regular PSR, ...



# Generalized Regular PSR Examples

GEC (model-based POMDP/PSR version):

$$d_{\text{GEC}} = \sum_{t=1}^{\infty} \gamma^t \sum_{h=0}^{\infty} \sum_{s=1}^{\infty} \gamma^{h+s} \left( P_{f^t}^{\exp(f^s; h)} \right)^{1/2};$$

where  $\exp(f^s; h) := f^s \circ h \text{Unif}(A) \circ h_{+1} \text{Unif}(U_{A;h+1})$  and  $U_{A;h+1} = A^{m-1}$  for  $m$ -step revealing/decodable POMDPs.

## Theorem (GEC of Generalized Regular PSR)

For  $\gamma$ -generalized regular PSR

$$d_{\text{GEC}} \leq \frac{d_{\text{PSR}} A^3 U_A^4 H}{4};$$

where  $d_{\text{PSR}}$  is the PSR rank and  $U_A = \max_{h \in [H]} |U_{A;h}|$ .

# Decision-Estimation Coefficient

DEC (Foster et al., 2021) is another complexity measure that is very general to cover most of the known tractable problems. We consider a set of models  $\mathcal{M}$  and Hellinger distance  $D_H^2$ :

$$\text{dec}(\mathcal{M}; \mathcal{M}_t) = \inf_{p_t} \sup_{\substack{M \in \mathcal{M} \\ \text{worst-case}}} \mathbb{E}_{z \sim p_t} [\underbrace{\text{Reg}_t^M(z)}_{\text{regret when } M \text{ is true model}} \mid \underbrace{D_H^2(M(z); \mathcal{M}_t(z))}_{\text{Easy to control}}];$$

- Convert our target (not easy to control) within one iteration to **something we know how to control** (assumption 4.1 of (Foster et al., 2021)):

$$\mathbb{E}_{z \sim p_t} \text{Reg}_t \leq \text{dec}(\mathcal{M}; \mathcal{M}_t) + \mathbb{E}_{z \sim p_t} D_H^2(M(z); \mathcal{M}_t(z));$$

where  $\mathcal{M}_t$  is a sequence of estimation and  $p_t$  is the solution in the definition of DEC.

- DEC is the **worst-case** cost for such a transformation from a game viewpoint and I think that is why DEC is also very close to the lower bound;
- We have

$$\mathbb{E} \text{Reg}(T) \leq \underbrace{\sum_{t=1}^T \text{dec}(\mathcal{M}; \mathcal{M}_t)}_{\text{Cost of transformation}} + \underbrace{\sum_{t=1}^T \mathbb{E}_{z \sim p_t} [D_H^2(M(z); \mathcal{M}_t(z))]}_{\text{New target: online learning}}; \quad (1)$$

# Decoupling Coefficient

Decoupling coefficient (Zhang, 2022; Agarwal and Zhang, 2022b,a) is a complexity measure that has applied to model-free/model-based RL and contextual bandit. We illustrate the main idea by the contextual bandit version. We consider a value class  $F = \{f : S \rightarrow [1, 1] \}$ :

$$E_{f^t, q^t; a^t = a^{f^t}(x^t)} \left[ \underbrace{V_{1; f^t}(x^t) - V_1(x^t; a^{f^t}(x^t))}_{\text{Feel-good regret}} \right] + \frac{d_{DC}}{4} + \underbrace{E_{a^t, q^t(a^t; x^t; S^t)} E_{f^t, q^t} [Q_{1; f^t}(x^t; a^t) - Q_1(x^t; a^t)]^2}_{\text{Easy to control}}$$

where we use  $a^{f^t}(x) := \operatorname{argmax}_{a \in \mathcal{A}} Q_{1; f^t}(x; a)$ . DC shares similar spirits with DEC but is different in:

- 1 Feel-good term:  $V_{1; f^t}(x^t; a^{f^t}(x^t)) - V_1(x^t; a^{f^t}(x^t))$ : we favor  $f$  with large value;
- 2 Flexible choice of **policy distribution**: suppose that  $f^t, q^t$ :
  - DC directly picks  $p^t(\cdot) := \operatorname{argmax}_{f \in \mathcal{H}} E_{f^t, q^t} [f(x^t)]$ ;
  - DEC solves the minimax problem of definition to get:

$$p^t(\cdot) = \operatorname{argmin}_{p^t(\cdot)} \sup_{f \in \mathcal{H}} E_{f^t, q^t} [ \underbrace{\operatorname{Reg}_{f^t}^M}_{\text{regret when } f \text{ is true model}} + \underbrace{D_H^2(f(\cdot); f^t(\cdot))}_{\text{Easy to control}} ]$$

- 3 Flexible choice of notion of new target.

# Reduction-based Framework

- GEC reduces out-of-sample  $V_{1;f^t}$  to **in-sample error estimation**:

- 1 A low GEC: model-based + model-free;
- 2 An effective in-sample error estimator;
- 3 Handle the difference between  $V_{1;f}$  and  $V_1$ ;

$$\text{Reg}(T) \lesssim d_{\text{GEC}} \sum_{t=1}^h \sum_{s=1}^{\lfloor \frac{1}{s} \rfloor} \mathbb{E} \left[ \sum_{i=1}^s (f^t)^i \right] + \frac{1}{h} d_{\text{GEC}} \sum_{t=1}^h \underbrace{\sum_{s=1}^{\lfloor \frac{1}{s} \rfloor} \mathbb{E} \left[ \sum_{i=1}^s (f^t)^i \right]}_{\text{New target: in-sample estimation}}$$

- DEC reduces out-of-sample  $V_1$  to **another out-of-sample target**:

- 1 A low DEC: model-based;
- 2 An effective online learning oracle;

$$\text{EReg}(T) \leq \underbrace{\sum_{t=1}^T \text{dec}^H(\mathcal{M}; t)}_{\text{Cost of transformation}} + \sum_{t=1}^T \mathbb{E} \left[ \rho^t \mathbb{E}_{\mathcal{M}_t} \left[ \sum_{j=1}^D \mathcal{M}_t(j, j) \right] \right]$$

New target: **online learning**

- DC/O-DEC reduces out-of-sample  $V_{1;f^t}$  to **another optimistic out-of-sample target**:

- 1 A low complexity measure: model-based + model-free;
- 2 An effective online learning oracle;
- 3 Handle the difference between  $V_{1;f}$  and  $V_1$ .

$$\text{EReg}(T) \leq \underbrace{\sum_{t=1}^T \text{odec}^D(\mathcal{M}; t)}_{\text{Cost of transformation}} + \sum_{t=1}^T \mathbb{E} \left[ \rho^t \mathbb{E}_{\mathcal{M}_t} \left[ \sum_{j=1}^D \mathcal{M}_t(j, j) \right] \right] + \sum_{t=1}^T \mathbb{E} \left[ \rho^t V_{1; \mathcal{M}_t}(x_1) \right]$$

New target: **online learning** with feel-good term